Analysis of Composite Beams using Method of Initial Functions

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Abstract: Method of initial functions is used for the analysis of composite laminated beams. The distribution of bending and shear stresses in composite laminated beams are different from beams of small thickness. The equations of two dimensional elasticity have been used for deriving governing equations. The order of the derived equations depends on the stage at which the series representing the stresses and displacements are truncated. No assumptions regarding physical behavior of beams are made. The beam theories which are based on assumptions are of a practical utility in the case of beams of moderate thickness. However in the case of thick or composite beams it becomes difficult to obtain useful results using theories based on assumptions.

Keywords: Composite Beams, Elasticity, Stresses, Initial Functions

1. Introduction

Beams that are built of more than one material are called composite beams. Examples are bimetallic beams, sandwich beams and reinforced concrete beams. Composite beams are widely used in many structures, as they are suitable for the development of lightweight structures. It is difficult to analyze the composite beams by the same bending theory we used for ordinary beams. The method of initial functions (MIF) has been used for the purpose of deriving the equations. The method of initial function (MIF) is an analytical method of elasticity theory. The method makes it possible to obtain exact solutions of different types of problems, i.e., solutions without the use of hypotheses about the character of stress and strain. Method of initial functions is used for the analysis of axially symmetric state of stress in elasto-dynamic problems [7]. According to this method, the basic desired functions are the displacements and stresses. Method of initial functions is used for two dimensional elasto dynamic problems for plain stress and plain strain conditions [2]. It is used for the analysis of thick circular plates. The governing equations are derived from the three-dimensional elasticity equations in cylindrical polar coordinates using Maclaurin series [8]. It is also used for the analysis of free vibration of rectangular beams of arbitrary depth. The frequency values are calculated using the Timoshenko beam theory and present the analysis for different values of Poisson's ratio [5]. MIF has been applied for deriving higher order theories for laminated composite thick rectangular plates [6]. They have used MIF for the static analysis of simply supported, orthotropic, and laminated circular cylindrical shell of revolution subjected to axisymmetric load. By using the continuity conditions of displacements and stresses on each interface between adjacent layers, the state equation for the laminate is obtained [1]. In the case of laminated beams it is quite difficult to assume a distribution of stresses and deflection with a fair amount of accuracy. It requires to developed simple models to explain this behavior as a function of material, geometry and loading parameters [9]. Developed governing equations for composite laminated deep beams by using method of initial functions. The beam theory developed can be used for beam sections of any depth [3]. Applied method of initial functions (MIF) for the analysis of orthotropic deep beams and compared the results with the available theory based on assumptions [4].

2. Problem Formulation

The laminated composite beam consists of N number of layers, serially numbered beginning from the bottom most layer. The thickness of any ith layer (i=1,……..N) is equal to hi and its elastic constants are \(E_{x,i}, E_{y,i}, G_{xy,i},\) and \(\mu_{xy,i} \).

Each layer has its own local coordinates system \(x_i\) and \(y_i\) (i=1,……..N) which is parallel to the Cartesian coordinate system \(x\) and \(y\) for the overall beam. The principal axes of orthotropy of each layer are parallel to the coordinate axes.

The constitutive relation for the material in the ith layer (i=1, ......N) are:

\[
\begin{align*}
\sigma_x &= C_{11,i} \varepsilon_x + C_{12,i} \varepsilon_y \\
\sigma_y &= C_{12,i} \varepsilon_x + C_{22,i} \varepsilon_y \\
\tau_{xy} &= C_{33,i} \gamma_{xy}
\end{align*}
\]

Where \(\sigma_x, \sigma_y\) and \(\tau_{xy}\) are bending, normal and shear stresses respectively.

And \(\varepsilon_x\) and \(\varepsilon_y\) are strains in \(x\) and \(y\) directions respectively.

The constants \(C_{11,i}\) to \(C_{33,i}\) expressed in terms of the elastic moduli of the material, for orthotropic material.

The equations of equilibrium for solids ignoring the body forces for two dimensional cases are:
\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad (4)
\]
\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \quad (5)
\]

For small displacements, the strain-displacement relations are:
\[
\varepsilon_x = \frac{\partial u}{\partial x} \quad (6)
\]
\[
\varepsilon_y = \frac{\partial v}{\partial y} \quad (7)
\]
\[
\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad (8)
\]

Where \( u \) and \( v \) are displacements in \( x \) and \( y \) directions. Local coordinate’s \( x_i \) and \( y_i \) for each layer lie in the same plane. Hence, the operators \( \alpha \) and \( \beta \) and the coordinates \( x \) and \( y \) need not have suffix \( i \). Stresses, strain and displacements can be expressed as functions of the global coordinates and need not have subscripts.

Eliminating \( \sigma_x \) from above equations the following equations are obtained, which can be written in matrix form as:
\[
\begin{bmatrix}
\hat{0} \\
\hat{v} \\
\hat{Y} \\
\hat{X}
\end{bmatrix} =
\begin{bmatrix}
0 & -\alpha & 0 & 1/G \\
C_{1x} & 0 & C_{2x} & 0 \\
0 & 0 & 0 & -\alpha \\
C_{3x} & C_{4x} & 0 & C_{5x} & 0
\end{bmatrix}
\begin{bmatrix}
u \\
\alpha \\
Y \\
X
\end{bmatrix}
\]
(9)

Where,
\[ X = \tau_{xy}, \quad Y = \sigma_y = C_{12} \varepsilon_x + C_{22} \varepsilon_y \]
\[ C_{1x} = -\frac{C_{11} \varepsilon_x}{C_{22}}, C_{2x} = \frac{1}{C_{22}}, C_{3x} = \frac{C_{12} \varepsilon_x}{C_{22}}, C_{5x} = C_{11} - C_{41} \]

Expressions for the constants \( C_{ij} \) to \( C_{55} \) are given in Appendix.

The equation (9) is written as:
\[
\frac{\partial}{\partial y} \left\{ \mathbf{S} \right\} = \left[ \mathbf{D}_T \right] \left\{ \mathbf{S}_i \right\} \quad (10)
\]

The solution of equation (10) is
\[
\left\{ \mathbf{S} \right\} = \left[ e^{\mathbf{D}_T y} \right] \left\{ \mathbf{S}_i \right\} \quad (11)
\]

Where \( \left\{ \mathbf{S}_i \right\} \) is the vector of initial functions, as the value of the state vector \( \left\{ \mathbf{S} \right\} \), at the bottom of the \( i \)-th layer i.e. at \( y_i = 0, \ (i = 1, \ldots, N) \).

Hence, \( \left\{ S_i \right\} = \begin{bmatrix} u_i, \ v_i, \ Y_i, \ X_i \end{bmatrix}^T \quad (12) \)

If \( u_i, \ v_i, \ Y_i, \ X_i \) are values of \( u, \ v, \ Y \) and \( X \) respectively, at the bottom plane of the \( i \)-th layer \( (i = 1, \ldots, N) \)

The solution of equation (11) can be written as:
\[
\left\{ \mathbf{S} \right\} = \left[ \mathbf{L}_i \right] \left\{ \mathbf{S}_i \right\} \quad (13)
\]

The equation (13) represents the general solution of two-dimensional problem for orthotropic materials.

Where \( \left[ \mathbf{L}_i \right] = e^{\mathbf{D}_i y} \quad (14) \)

The transfer matrix \( \left[ \mathbf{L}_i \right] \) relates the stresses and displacements at the bottom plane of the \( i \)-th layer to the same at any other parallel plane within the same layer \( (i = 1, \ldots, N) \).

It is a square matrix of the form
\[
\left[ \mathbf{L}_i \right] =
\begin{bmatrix}
L_{uu} & L_{uv} & L_{uy} & L_{ux} \\
L_{vu} & L_{vv} & L_{vy} & L_{vx} \\
L_{yu} & L_{vy} & L_{yy} & L_{yx} \\
L_{yu} & L_{vy} & L_{yy} & L_{yx}
\end{bmatrix}
\]
(15)

Expanding (10) in the form of a series
\[
\left[ \mathbf{L}_i \right] = \sum_{j=0}^{\infty} \frac{y_j^j}{j!} \left[ \mathbf{D}_i \right]^j \quad (16)
\]

Where, \( \left[ \mathbf{I} \right] \) is a unit matrix.

The truncation of series (16) depend on the order of the beam theory desired.

In the case of a layered composite beam loaded at the top surface, the state of stresses and displacements at the free bottom surface of the beam is given by:
\[
\left\{ \mathbf{S}_i \right\} = \begin{bmatrix} u_i, \ v_i, \ 0, \ 0 \end{bmatrix}^T \quad (17)
\]

Let
\[
\left\{ \mathbf{S}_T \right\} = \begin{bmatrix} u_T, \ v_T, \ Y_T, \ X_T \end{bmatrix}^T \quad (18)
\]

Where \( u_T, \ v_T, \ Y_T, \ X_T \) are the values of stresses and displacements at the top surface of the layered beam.

Relating the stresses and displacements at the top surface of the layer to those at the bottom surface by successive application of the transfer matrix \( \left[ \mathbf{L}_i \right] \) across each layer, one obtains:
\[
\left\{ \mathbf{S}_T \right\} = \left[ \mathbf{A} \right] \left\{ \mathbf{S}_i \right\} \quad (19)
\]

Where,
\[
\left[ \mathbf{A} \right] = \left[ \mathbf{L}_N \right]_{y_N = h_N} \cdots \left[ \mathbf{L}_2 \right]_{y_2 = h_2} \left[ \mathbf{L}_1 \right]_{y_1 = h} \quad (20)
\]

The terms of the matrix \( \left[ \mathbf{A} \right] \) are evaluated after expanding the exponential in the form of a series.

The matrix has a form:
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\[
[A] = \begin{bmatrix}
A_{uu} & A_{uv} & A_{uy} & A_{ux} \\
A_{vu} & A_{vv} & A_{vy} & A_{vx} \\
A_{yu} & A_{vy} & A_{yy} & A_{yx} \\
A_{yu} & A_{vy} & A_{yx} & A_{xx}
\end{bmatrix}
\]  (21)

The equation (19) relates the boundary conditions at the top surface to those at the bottom surface and is useful for deriving governing differential equations for a layered beam having a particular number of layers.

3. Application to the problem of composite beam having three layers

A composite beam consists of the three layers. Therefore the matrix \([A]\) becomes

\[
[A] = \begin{bmatrix}
L_3 & \cdots & L_2 & \cdots & L_1
\end{bmatrix}
\]  (22)

Where \(h_1, h_2, \text{ and } h_3\) are the thickness of three layers.

The conditions at top are given by:

\[
\{ S_T \} = \begin{bmatrix}
u_T \\
v_T \\
- p \\
0
\end{bmatrix}
\]  (23)

Substituting the expressions (17) and (21) in the equations (19) we get:

\[
A_{Xu} u_1 + A_{Xv} v_1 = 0
\]  (24)

\[
A_{Yu} u_1 + A_{Yv} v_1 = - p
\]  (25)

These equations are exactly satisfied by

\[
u_1 = A_{Xv} \phi
\]  (26)

\[
v_1 = A_{Xu} \phi
\]  (27)

Where \(\phi\) is an unknown auxiliary function substituting the value of \(u_1\) and \(v_1\) from the equations (26) and (27) in the equation (25), the differential equation governing the problem of a normally loaded composite beam is obtained:

\[
(A_{Yu} \cdot A_{Xv} - A_{Yv} \cdot A_{Xu}) \phi = - p
\]  (28)

The order of the governing differential equation (28) depends on the order of the terms in the matrix \([A]\).

The auxiliary function \(\phi\) is chosen such that it satisfies the governing differential equation (28), as well as the boundary conditions at the edges of the beam. Initial functions are obtained from equations (26) and (27). By operating on the initial functions by the transfer matrix \([L]\) successively across each layer, we can determine the stresses and displacements, within the entire beam.

4. Analysis of composite Beam

Consider a composite beam having three layers of thickness \(h_1, h_2, \text{ and } h_3\) of orthotropic material.

The following values of beam dimensions are chosen for the particular problem,

\(H = 200\) cms, \(l = 400\) cms

\(h_1 = 20\) cms, \(h_2 = 160\) cms and \(h_3 = 20\) cms.

The following material properties are taken:

\(E_x = 0.10 \times 10^5\) N/mm\(^2\), \(E_y = 0.05 \times 10^5\) N/mm\(^2\)

\(\mu_{xy} = 0.25, \mu_{yx} = 0.06, G = 0.10 \times 10^5\) N/mm\(^2\)

The boundary conditions of the simply supported edges are:

\[X = Y = v = 0, \text{ at } x = 0 \text{ and } x = l\]

The boundary conditions are exactly satisfied by the auxiliary function.

\(\phi = A_1 \sin (\pi x/l)\)

A sinusoidal normal loading is assumed, on the top surface of the beam:

\(P(x) = - P_0 \sin (\pi x/l)\)

Taking \(P_0 = 100\) kg/cm and \(x = l/2\)

Initial functions obtained are

\(u_1 = 4.7679e^{-04}\)

\(v_2 = 0.0030\)

The values of \(u_1\) and \(v_1\) are substituted in expression (13) for obtaining the values of stresses and displacements at top of the composite beam.

\(Y = -100.0\)

\(X = 0\)

5. Conclusion

The initial functions are operated upon by the transfer matrix successively across each layer until the entire beam is analysed and the stresses at the top surface are again obtained. The stresses evaluated at top surface are quite close in value to the intensities of the corresponding applied loads.

Governing equation of desired order according to the requirements of a beam problem of any specific thickness and material property is obtained using MIF.

MIF gives accurate results in case of small thickness, large thickness and layered members. The MIF assumes a significant importance in the analysis of thick, composite or sandwich beams.

6. Notation

\(l\) - Span of beam
\(E\) - Young’s modulus of Elasticity
\(G\) - Shear modulus of Elasticity
\(\mu\) - Poisson’s ratio
7. Appendix

\[ C'_{11} = \frac{E_x}{1 - \mu_{xy}\mu_{yx}}, \]

\[ C'_{12} = \frac{E_y\mu_{xy}}{1 - \mu_{xy}\mu_{yx}}, \]

\[ C'_{22} = \frac{E_y}{1 - \mu_{xy}\mu_{yx}}, \]

\[ C'_{33} = G_{xy}, \]

\[ \mu_{xy} = \mu_{yx} \frac{E_x}{E_y}. \]

8. References


